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Analytical solution to the Black-Scholes Equation: Adomian Decomposition Method Versus Lie Algebraic Approach

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Abstract

In this paper, we compare two relevant methods to find Analytical solution of the Black-Scholes Equation. First, we apply the Adomian Decomposition Method as in [2], to obtain a solution to the aforementioned equation with boundary condition for a European option. Secondly, we apply the Lie algebraic Approach for determining the solution as in [7]. Those two methods conducted us to investigate the thin line between the underlying results. Finally, we suggest a simple enhanced Due Diligence on both approaches.

***Keywords**: Black-Scholes equation, Adomian Decomposition Method, Lie Algebraic Approach, Symmetry analysis*

INTRODUCTION

The Black-Scholes equation, which was first published it in 1973 by Fischer Black and Myron Scholes[5], has been created to present a general equilibrium of financial option pricing by the means of stochastic differential Equation.

This paper presents the Adomian Decomposition Method (ADM) applied to a diffusion equation [2], with non-null Dirichlet boundary conditions, obtained after reducing the Black-Scholes Equation with homogenous boundary equations. The method is based on both the decomposition of the unknown function in an infinite series $\sum_{n=0}^{\infty} A_n$ and the decomposition of the nonlinear term of the equation in another series $\sum_{n=0}^{\infty} A_n$, when the A_n are named Adomian polynomials. The method has been applied in many deterministic and stochastic problems. The linear and nonlinear problem has been used from physics to Finance and vice versa.

In general, the most effective strategy regarding option pricing can be found by solving Black-Scholes equation using the Lie Symmetry theory [1]. This theory which has been widely adopted in physics and engineering were applied for the first time in finance by Gazizov and Ibragimov[3]. They implemented a relevant technic to transform the Black-Scholes equation into a classical heat equation.

To the best of our knowledge, there is a lack of research on the comparison between the Adomian Decomposition Method and the Lie Algebraic Approach. In this paper, we explore Both Approaches to the pricing of European Call options problems in risky, flexible and irreversible financial ecosystems.

MATERIALS AND METHODS

Black- Scholes equation

In financial mathematics, it can be demonstrated that by studying a strategy of self-financing, one can reach the following partial differential equation called Black-Scholes equation:

$$rf(t,x) = f_t(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) + rxf_x(t,x) \quad x > 0 \ , t \in [0,T] \qquad (1)$$

Where r represent the value of the action t the time, f the option price, r is the type of interest of the market of debt, σ the volatilily of the action measured as the standard deviation of the logarithm of the value of the action. In this paper, we give an analytic solution of call option problem,

$$\begin{cases} rc(t,x) = C_t(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) + rxC_x(t,x) \;\; x > 0, \;\; t \in [0,T] \\ C(T,x) = \max(x-K,0) \\ C(t,x) = x - Ke^{-r(T-t)} \quad where \;\; x \to \infty \\ C(t,0) = 0 \qquad \forall\, t > 0 \end{cases} \qquad (2)$$

To reduce (2) into a diffusion problem, we use change of variable given by

$$\tau = \frac{1}{2}\sigma^2(T-t), \;\; y = \ln\left(\frac{X}{K}\right), \gamma = \frac{2r}{\sigma^2}$$

And we assume that (t,x) can be expressed by:

$$C(t,x) = Ke^{-ay-b\tau}U(\tau,y)$$

where

$$a = \frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right) \; and \; b = (1+a)^2$$

Thus, (2) transforms into

$$\begin{cases} u_t(\tau, y) = u_{xx}(\tau, y), \; y > 0, \quad \tau \in \left[0, \dfrac{\sigma^2 T}{2}\right] \\ u(0, y) = \max\left(e^{\frac{1}{2}(\gamma+1)y} - e^{\frac{1}{2}(\gamma-1)y}, 0\right) \\ u(\tau, L) = e^{\frac{1}{2}(\gamma+1)L + \frac{1}{4}(\gamma+1)^2\tau} - e^{\frac{1}{2}(\gamma-1)L + \frac{1}{4}(\gamma-1)^2\tau} \\ u(\tau, 0) = 0 \end{cases} \quad (3)$$

The general solution of (3) is given by:

$$C(t, x) = Ke^{-\frac{1}{2}(\gamma+1)x - \frac{1}{4}(\gamma-1)^2\tau} U(\tau, y) \quad (4)$$

This to obtain a solution of call option problem (4), we reduce equation (2) into (3). This is, we have reduced the Black-Scholes equations into a diffusion equation in vide to use all given Boundary conditions.

Adomian Decomposition Method versus Lie Algebraic Approach

Adomian Decomposition Method(ADM)

Given a differential equation

$$F\,u(t) = g(t) \quad (5)$$

Where F represents a non lineare differentiel operator which includes both lineare and non lineare terms, so that equation (5) can be written as

$$Lu(t) + Ru(A) + Nu(t) = g(t)$$

Where

$L + R$ *is the linear operator*,

L *is an easily invertible operator*,

R *is the remainde linear operator*,

N *represents the non linear operator*,

g *is the independant fonction of* $u(t)$.

Resolving for $Lu(t)$,

$$Lu(t) = g(t) - Ru(t) - Nu(t)$$

Since L is invertible, we have that:

$$L^{-1}Lu(t) = L^{-1}g(t) - L^{-1}Ru(t) - L^{-1}Nu(t)$$

A_n Equivalent expression

$$u(t) = \varphi + L^{-1}g(t) - L^{-1}Ru(t) - L^{-1}Nu(t) \quad (6)$$

Where φ is the integrale constant and satisfies $\varphi = 0$. For problems with an initial value in $t = a$, we have convenitly defined L^{-1} for $L = \frac{d^n y}{dx^n}$,

Which is the definite integral of a to t. This method assumes a solution in the form of an infinite series for the unknown function $u(t)$ given by,

$$u(t) = \sum_{i=0}^{\infty} u_i(t) \quad (7)$$

The nonlinear term $Nu(t)$ is decomposed as:

$$Nu(t) = \sum_{n=0}^{\infty} A_n \, (u_0, u_1, \dots, u_n) \quad (8)$$

Where A_n is called Adomian polynomial, and depends an the particularity of the nonlinear operator. The A_n are calculated in general way by the following formula:

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left(\sum_{j=0}^{\infty} \lambda^j u_j\right) \Big/ {}_{\lambda=0} \quad (9)$$

Can be solved using software such as Maple[4] substituting (7) and (8) in (6) we have,

$$\sum_{i=0}^{\infty} u_i(t) = \varphi + L^{-1} g(t) - L^{-1} R \sum_{i=0}^{\infty} u_i(t) - L^{-1} \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$$

And thus a solution is obtained by

$$\begin{cases} u_0(t) = \varphi + L^{-1} g \\ u_{n+1}(t) = -L^{-1} R u_n(t) - L^{-1} A_n(u_0, u_1, \dots, u_n) \end{cases}$$

The approximations are given by

$$\psi_k = \sum_{i=0}^{k-1} u_i(t)$$

The decomposition of the solution series converges in general very quickly. This nears that few terms are required for the approximation convergence of this method has been rigorously established in[9].

Lie Algebraic Method

Symmetry analysis is of the most powerful analytical techniques for systematically solving PDEs through transformation. According to Lie[4], differential equations can be integrated or reduced to lower order terms based or synthesis symmetry. Baumann[1] found that two independent variables x, y,and are dependent variable v can be solved using the below equation:

$$\Delta\left(t, x, v, \frac{\partial v}{\partial t}, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial t}\right) = 0 \quad (10)$$

Which illustrate is parameter of the Lie group of transformations

$\tilde{t} = T_t(t, x, v, \varepsilon)$

$\tilde{x} = T_x(t, x, v, \varepsilon)$ (11)

$\tilde{v} = T_v(t, x, v, \varepsilon)$

If we assume the solution of (10) is $v = \theta(t, x)$, the transformations can be expressed as follow:

$\tilde{t} = T_t(t, x, \theta(t, x); \varepsilon)$

$\tilde{x} = T_x(t, x, \theta(t, x); \varepsilon)$ (12)

$\tilde{v} = T_v(t, x, \theta(t, x); \varepsilon)$

Therefore $\tilde{v}$ is the solution to the transformations this unique problem can be solved by using the functional equation:

$\theta = \big(T_t(t, x, \theta(t, x); \varepsilon), T_x(t, x, \theta(t, x); \varepsilon)\big) = \tilde{v}$ (13)

By replacing the transformations in (12) with infinitesimal representations, we obtain the following new equation:

$\tilde{t} = t + \varepsilon\xi_1(t, x, v; \varepsilon)$

$\tilde{x} = x + \varepsilon\xi_2(t, x, v; \varepsilon)$ (14)

$\tilde{v} = v + \varepsilon\phi_1(t, x, v; \varepsilon)$

Then (13) can be simplified with infinitesimal representations with ε as the group parameter shown below:

$\theta + \varepsilon\phi_1(t, x, \theta; \varepsilon) = \theta(x + \varepsilon\xi_2(t, x, \theta; \varepsilon) + \varepsilon\xi_1(t, x, \theta; \varepsilon))$ (15)

According to the Taylor expansion, we can subtract the left-hand side from the right-hand side of (15) to yield $\varepsilon = 0$ as shown

$\xi_1 \frac{\partial\theta}{\partial t} + \xi_2 \frac{\partial\theta}{\partial X} - \phi_1 = 0$ (16)

Thus, the invariant surface condition of (16) can be determined as follows!

$\vec{v} \cdot F(t, x, v) = \xi_1 \frac{\partial F}{\partial t} + \xi_2 \frac{\partial F}{\partial X} - \phi_1 \frac{\partial F}{\partial v} = 0$ (17)

Where the tangent vector $\vec{v}$ is calculated in the equation as shown:

$\vec{v} = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial X} + \phi_1 \frac{\partial}{\partial v}$ (18)

We can solve the invariant condition of first order PDEs with the unit vector ξ_1, ξ_2 and ϕ_1 by using the following characteristics of differential equations:

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon} = \xi_1(\tilde{t}, \tilde{x}, \tilde{v})$$

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon} = \xi_2(\tilde{t}, \tilde{x}, \tilde{v}) \qquad (19)$$

$$\frac{\mathrm{d}\tilde{\tilde{v}}}{\mathrm{d}\varepsilon} = \phi_1\,(\tilde{t}, \tilde{x}, \tilde{v})$$

Where the initial conditions are as follows:

$$\left.\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = t$$

$$\left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = x \qquad (20)$$

$$\left.\frac{\mathrm{d}\tilde{\tilde{v}}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = v$$

After reducing (10), the PDEs can be solved analytically via the characteristic curves.

RESULTS AND DISCUSSION

Solution of the Black-Scholes Equation

ADM application for European option

Given (3), following the ADM procedure, Considering:

$L = \frac{\mathrm{d}\mu}{d\tau}, R = \frac{d^2u}{dx^2}, N = 0, yg = 0$

We obtain,

$$L^{-1}u_t(\tau, y) = L^{-1}\mu_{xx}(\tau, y)$$

$$u(\tau, y) = u(0, y) + \int_0^\tau u_{xx}(s, y)\,\mathrm{d}S$$

Assuming a solution in the form of an infinite series:

$$u(x, y) = \sum_{i=0}^{\infty} \mu_i\,(x, y)$$

We now have,

$$\sum_{i=0}^{k} \mu_i(\tau, y) = u(0, y) + \int_0^\tau \sum_{i=0}^{\infty} u_{i_{xx}}\,(s, y)\,\mathrm{d}S$$

For an approximation up to (k +1) terms, we have

$\sum_{i=0}^{k} \mu_i(\tau, y) = u(0, y) + \int_0^\tau \sum_{i=0}^{\infty} u_{i_{xx}}\,(s, y)\,\mathrm{d}S,$

$$\leftrightarrow \sum_{i=0}^{k} \mu_i(\tau, y) = u(0, y) + \sum_{i=0}^{k} \int_0^{\tau} u_{i_{xx}}(s, y)\, \mathrm{d}S$$

Thus, the $(\mathrm{k}+1)$ $-$ th approximation for the solution is given by

$$\psi_k = \sum_{i=0}^{k-1} \mu_i(t) \approx u(t)$$

And so, the solution of (3) is determined, for the call option, by

$$C(t, x) = K\mathrm{e}^{-1/2(\gamma+1)x - 1/4(\gamma-1)^2\tau}\, \psi_k$$

Lie Algebraic approach application for European option

We utilize the prolongation formula discussed in [6] to obtain the characteristic differentials associated with the European call option equation (2), which is given as follows:

$\xi_2 - (q-x)(-4(\xi_2)_x + 2(\xi_1)_t + (q-x)(2(r-\mu)(\xi_1)_x + (q-x)\sigma^2 (\xi_1)_{xx})) =$
0 (21)

Before going further, let recall (2), as we assume that $C(T, x) = \max(x-k, 0)$:

$$rC(t, x) = C_t(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x) + rxC_x(t, x)$$

$$rC(t, x) = C_t(t, x) + rxC_x(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x)$$

$C_t(t, x) + rxC_x(t, x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t, x) = 0$ (22)

If we denote $x = q - x$ and $r = r - \mu$

(22) Can now take the following format,

(23) $C_t(t, x) + (r-\mu)(q-x)C_x(t, x) + \frac{1}{2}\sigma^2 (q-t)^2 C_{xx}(t, x) = 0$

We can now make all the necessary computations on (21):

(21) Implies

$$(q-x)(-2(r-\mu)(\xi_2)_x + 2(r-\mu)(\xi_1)_t + (q-x)(2(r-\mu)^2(\xi_1)_x - \sigma^2 ((\xi_2)_{xx} - (q-x)(r-\mu)(\xi_1)_{xx}) - 2(\phi_1)_{xv}))) - 2((r-\mu)\xi_2 + (\xi_2)_t) = 0$$

$$2(\phi_1)_t + (q-x)(2(r-\mu)(\phi_1)_x + (q-x)^2\sigma^2(\phi_1)_{xx}) = 0$$

$$2((\xi_2)_v - (q-x)^2\sigma^2(\xi_1)_{xv} = 0$$

$2(\xi_2)_{xv} - 2(r-\mu)(q-x)(\xi_1)_{xv} - (\phi_1)_{vv} = 0$

$$(\xi_2)_{xv} - (r-\mu)(q-x)(\xi_1)_{xv} = 0$$

$$(\xi_1)_x = (\xi_1)_{vv} = (\xi_1)_v = 0$$

The solutions are clearly illustrated in the equation below:

$$\xi_1 = a_1 + t\left(a_2 + \frac{8a_3 t\sigma^2}{(2r - 2\mu + \sigma^2)^2}\right)$$

$$\xi_2 = \frac{1}{4}\left((q - x)(-4a + 2at(r - u) + (a_2 - 4a_5)t\sigma^2)\right.$$
$$\left. + 2(x - q)\left(a_2 + \frac{16a_3 t\sigma^2}{(2r - 2u + \sigma^2)^2} ln(x - q)\right)\right)$$

$$\phi_1 = C\left[a_6 + a_3 t^2 + a_5 \ln(x - q) + \frac{4a_3(\ln(x-q))^2}{(2r-2\mu+\sigma^2)^2} + t\frac{-8a_3\sigma^2 + (2r-2\mu+\sigma^2)(a_5(2r-2\mu+\sigma^2)^2 + 8a_3 \ln(x-q))}{2(2r-2\mu+\sigma^2)^2}\right]$$

Where $C \equiv C(t, x)$

Where a_i for i from $1 - 6$ become arbitrary constants.

They also provide the infinite-dimensional Vector Space for infinitesimal symmetries of Equation (23), including the following Operator

$$V_1 = \partial_t$$
$$V_2 = t\partial_t + k_2\partial_x + k_3 x\partial_x + k_3 \ln(x - q)\,\partial_x + k_5 \ln(x - q)x\,\partial_x$$
$$V_3 = \frac{1}{k_1}(\,k_6 t^2\partial_t + k_7\,\partial_c + k_8 \ln(x - q)\,\partial_x + k_9 \ln(x - q)x\,\partial_x$$
$$V_4 = k_{10}\,\partial_x$$
$$V_5 = k_{11}\,\partial_c + k_{12}\,\partial_x + k_{13}\,\partial_x$$
$$V_6 = C\partial_c$$

Where k_i for $1 \le i \le 6$ are defined by $r, \mu, \sigma, q, t.$

Also, the symmetry algebra is calculated using $V_1 - V_6$ as to establish an invariant solution equation for (23).

If $v = F(t, \eta)$, in the scenario we have $v = \phi_1(t, x, v, \varepsilon)|_{\varepsilon\to 0}$

By assuming the value of:

$a_1 - a_3$, C_1 and C_2 ($a_2 = a_3 = a_5 = 0,\ a_1 = C_1,\ a_4 = C_2\ and\ a_6 = 1$),

Then $\xi_1, \xi_2\ and\ \phi_1$ can be determined

$$\xi_1 = c_1, \xi_2 = c_2(q - x), \phi_1 = C\ where\ C \equiv C(t, x)$$

After finding the relationship between x and t, the invariant under the symmetry group of $V_1 - V_6$ can be calculated using the following equation:

$inv = \frac{C_2 t}{C_1 \ln(q-x)}$ For $C_1, C_2 \neq 0$

Because of (23) $\xi = t - \frac{C_1}{C_2}\ln(q-x)$ and the group invariant solution of (23), we have:

$$\sigma^2 - C^2(2(r-\mu) + \sigma^2))F(\xi) + (2C_2(C_2 + C_1(r-\mu)) + \left(2C_2(C_2 + C_1(r-C)\right) + C_1(C_2-2)\sigma^2)F'(\xi) + C_1^2\sigma^2 F''(\xi) = 0$$

Where,

$F(\xi) = (q-x)^{-\frac{1}{C_2}}\mathrm{C}(\mathrm{t},\mathrm{x})$

The $F(\xi)$ is the function with the arbitrary Constant ω_1 and ω_2:

$$F(\xi) = \omega_1 e^{g_1\xi} + \omega_2 e^{g_2\xi}$$

Where,

$$g_1 = \frac{-2C_2\left(C_2 + C_1(r-\mu)\right) + C_1(C_2-2)\sigma^2}{2C_1^2\sigma^2} + \frac{\sqrt{C_2^2(4\left(C_2 + C_1(r-\mu)\right)^2 + 4C_1(C_2 - 2 + C_1(r-\mu)\sigma^2 + C_1^2\sigma^2}}{2C_1^2\sigma^2}$$

$$g_2 = \frac{-2C_2\left(C_2 + C_1(r-\mu)\right) + C_1(C_2-2)\sigma^2}{2C_1^2\sigma^2} - \frac{\sqrt{C_2^2(4\left(C_2 + C_1(r-\mu)\right)^2 + 4C_1(C_2 - 2 + C_1(r-\mu)\sigma^2 + C_1^2\sigma^2}}{2C_1^2\sigma^2}$$

The invariant solution is

$C(t,x) = \omega_1 (q-x)^{\frac{1-C_1 g_1}{C_2}} e^{g_1 t} + \omega_2 (q-x)^{\frac{1-C_1 g_2}{C_2}} e^{g_2 t}$ (24)

We clearly see that, despite the fact ADM method is fast, Lie algebraic Approach provides more rough and detailed results.

CONCLUSION

Since the Adomian Decomposition Method (ADM) converges quickly as shown in [9], it turns out to be an efficient alternative tool to solve the Black-Scholes equation. In general, both ADM and Lie Algebraic Approach give an analytical solution for Partial Differential equations problems, without implying that this solution is adequate to a given problem, because it doesn't use all boundary conditions. But we have clearly notice the roughness of the Lie Algebraic Approach which provide more accurate and detailed solution. Therefore, comparing both approaches may be very useful for practitioners.

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